1.9 Arc Length & Curvature

Thursday, February 4, 2021 12:14 PM

Arclength & curvature

ie. intrinsic properties of curves

e.g. circle of radius 1 given by

$(x, y, z) = (\cos t, \sin t, 0)$

or $(x, y, z) = (\cos t^2, \sin t^2, 0)$

these are 2 parametrizations of the same curve.

Q What abt $(x, y, z) = (\cos t^2, \sin t, 0)$?

A No, it's a completely different curve

Another eg

parabola in xy plane $(x, y, z) = (t, t^2, 0)$

how about $(x, y, z) = (t^3, t^6, 0)$

→ also same parabola

What about $(x, y, t) = (t^2, t^6, 0)$?

→ not just a diff parametrization of the same curve

→ this looks like $y = x^3$

In general say we have

$\vec{F}(t) = (x(t), y(t), z(t))$

and let g be a single-var function

Such that: For $t \in D'$ (possibly some other domain)

① $g(t) \in D$

② g is strictly monotone increasing, ie,

for $t_1, t_2 \in D'$ if $t_1 < t_2$

then $g(t_1) < g(t_2)$

eg $g(t) = t^2$ is monotone increasing on $D' = [0, \infty)$

then $\vec{F}(g(t))$ is vector-valued func defined for $t \in D'$ (b/c then $g(t) \in D$ so we can give it as input to $\vec{F}$.

now $\vec{F}(t)$ and $\vec{F}(g(t))$ are parameterizations of the same curve

Recall given $\vec{F}(t)$,

velocity $\vec{F}'(t)$

acceleration $\vec{F}''(t)$

speed $\|\vec{F}'(t)\|$

suppose $\vec{F}(t)$ is defined on $[a, b]$

(i.e $[a, b] \subseteq D$)

Define a func s as follows:

for $t \in [a, b]$, let $s(t)$ denote

Define a func s as follows:

for $t \in [a,b]$, let $s(t)$ denote the distance the obj has traveled since $t=a$

So $S(t)$ is a nonnegative real #.

Q What is $s(a)$?

A As t goes from a to b, $s(t)$ generally increases as long as the obj isn't stationary, ie as long as $\vec{F}'(t) \neq 0$.
[In that case, s is monotonically increasing]
(More generally, if $\vec{F}'(t) = 0$ at a single point but is nonzero everywhere else, then monot incr.)

Qualititatively, we know

① $s(a) = 0$
② s increases (or stays same if $\vec{F}(t) = 0$) as t goes from a to b.

Q Quantitatively, how to compare s?

A deriv ds/dt
is how much speed change/time
$$\frac{ds}{dt} = ||\vec{F}(t)||$$

So we know:
① $s(a) = 0$
② $\frac{ds}{dt} = ||\vec{F}'(t)||$

Using FTC,
$$s(t_1) = \int_a^{t_1} ||\vec{F}'(t)|| dt$$

To compute $s(t)$, you have to
① compute a derivative
② find a magnitude (as fcn of t)
③ compute a single-var integral

e.g. $\vec{F}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$
(circle of radius 1)
① $\vec{F}'(t) = -\sin(t)\vec{i} + \cos(t)\vec{j}$
② $||F'(t)|| = \sqrt{(-\sin t)^2 + \cos^2 t}$
$= \sqrt{\sin^2 t \cos^2 t} = 1$

③ Length from a to b

③ Length from a to b

$$s(b) = \int_a^b 1\,dt = b - a$$

=> length of sector of circle of radius 1 is the angle (in radians) of the sector.

* see book for helix *

<u>Another way to write arc length</u>

$$s(t_1) = \int_a^{t_1} \|\vec{f}\,'(t)\|\,dt$$

why "t_1"?

(Compare in single var: $F(t_1) = \int_a^{t_1} f(t)\,dt$)

$$= \int_a^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_a^{t_1} \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right](dt^2)}$$

$$= \int_a^{t_1} \sqrt{dx^2 + dy^2 + dz^2}$$

"intrinsic form"

→ independent of parameterization

<u>Can prove</u> that s is independent of parameterization using chain rule

<u>Suppose</u> $\vec{f}(t)$ defined for

$t \in [a,b] \subseteq D$.

Say $[c,d] \subseteq D'$ and g maps D' to D.

(and is strictly monot. incr)

suppose $g(c)=a$
and $g(d)=b$

"the interval $[c,d]$ corresponds under change of parameterization to $[a,b]$

!!!

1st parameterization $\vec{F}'(t)$ $t \in D$
2nd parameterization $\vec{F}(g(t))$ $t \in D$

$t=a$ in 1st param corresponds to
$t=c$ in 2nd
i.e. $\vec{f}(a)=\vec{f}(g(c))$

Q What do we mean when say arc length independent of parameterization.

A We mean that

$$\int_a^b \|\vec{f}'(t)\|\,dt = \int_c^d \|(\vec{f}\circ g)'(t)\|\,dt$$

Q Why?

A Follows by chain rule (integration by substitution)

Arc Length Parameterization

Recall As long $\vec{F}(t)$ doesn't remain const for period of time, $s(t)$ is strictly monotonically increasing in t.

$\Rightarrow$ can use it for reparametrization

choose g to be inverse fcn of s ← relative to some initial pt $t=a$
i.e. $g(s(t))=t$

Now

Consider $\vec{F}\circ g$ and input arclength, then we get corresponding $\vec{f}$.

e.g. circle $(\cos t, \sin t, 0) = \vec{f}(t)$
We know arclength from 0 is t.

Arclength from $t=a$ is $t-a$.
so $s(t) = t-a$
$\Rightarrow g(t) = t+a$
(just as $s(a)=0$, also $g(0)=a$)

<u>Try</u> circle radius 2
$\vec{f}(t) = (2\cos t, 2\sin t, 0)$

<u>Now</u>

$s(t) = 2(t-a)$

<u>What's $g(t)$?</u>

$g(t) = t/2 + a$

(another way to write: $t = \frac{s}{2} + a$)

<u>Now</u> What's $\vec{F}(g(t))$?

<u>A</u> $(2\cos(g(t)), 2\sin(g(t)), 0)$
$= (2\cos(\frac{t}{2}+a), 2\sin(\frac{t}{2}+a), 0)$

this is the parametrization by arc length starting at a.

<u>ie</u> $t=a$ in og parametrization corresponds to $t=0$ in the arc length parametrization

<u>Cylindrical coords</u>

<u>Recall</u> $x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad z=z$

And $s = \int \sqrt{dx^2 + dy^2 + dz^2}$

(another way to write: $ds^2 = dx^2 + dy^2 + dz^2$)

<u>Idea</u> Apply d to both sides of $x = r\cos\theta$
$\Rightarrow dx = d(r\cos\theta) = (dr)(\cos\theta) + r(d\cos\theta)$

<u>and</u> $d\cos\theta = d\theta\left(\frac{d\cos\theta}{d\theta}\right) = -\sin\theta\, d\theta$

so $dx = dr(\cos\theta) + r(-\sin\theta\, d\theta)$
$= \cos\theta\, dr - r\sin\theta\, d\theta$

$dy = d(r\sin\theta) = \sin\theta\, dr + r\, d\sin\theta$

$$= \sin\theta\, dr + r\cos\theta\, d\theta$$

$$\Rightarrow dx^2 + dy^2$$
$$= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2$$
$$= \cos^2\theta\, dr^2 + \sin^2\theta\, dr^2 + r^2\sin^2\theta\, d\theta^2 + r^2\cos^2\theta\, d\theta$$
$$= dr^2 + r^2 d\theta^2$$

$$\Rightarrow dx^2 + dy^2 + dz^2 = 0$$
$$= dr^2 + r^2 d\theta^2 + dz^2$$
$$\Rightarrow s = \int \sqrt{dx^2 + dy^2 + dz^2}$$

intuitive/abstract

$$= \int \sqrt{dr^2 + r^2 d\theta + dz^2}$$

For computation

in terms of t $$= \int \sqrt{\left(\frac{dr}{dt}dt\right)^2 + r^2\left(\frac{d\theta}{dt}dt\right)^2 + \left(\frac{dz}{dt}\right)dt^2}$$

Read [CH] for next time

$$= \int dt \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

What is curvature?

It should be a quantity describing qualitative visual property of how much a path is curved.

Eg — 0 for a line (iff $\varnothing$ curvature)

— non-zero for circle

- small for large radius circle
- large for small radius circle

curvature = "dizziness factor"

Notice - a curve is a line iff curvature $= \varnothing$

- $y = f(x)$ is a line iff 2^{nd} derivative $f''(x) = 0$

Q Is curvature just like 2^{nd} deriv?
(eg if $(x, y) = (t, f(t))$)

A No.

eg $y = x^3$ then for x large,

$\frac{d^2y}{dx^2}$ is large, but curvature is small